Chapter Nine.

Piecewise defined relationships.

For each of the three situations given below choose the one graph from the six shown that best fits the situation. Then, for each situation, having chosen the most appropriate graph, make a sketch of the graph and include labels and numbers on each axis. (If you think that none of the graphs fit the situation draw your own appropriate graph.)

Situation One.

The income tax system in Australia is what is known as a "progressive system". This means that the *rate* of income tax increases as a person's taxable income increases.
For this situation suppose that the following progressive system were to apply:

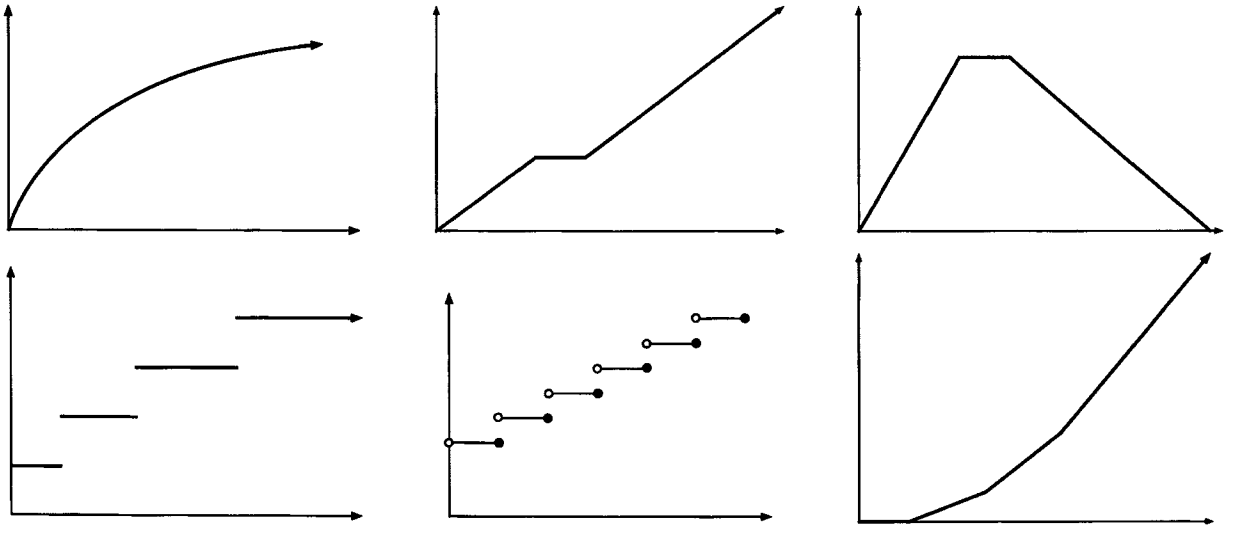
Taxable income	Rate at which tax is deducted.
\$0 to \$20000	Nil
\$20001 to \$50000	20% of every dollar over \$20000.
\$50001 to \$80000	\$6000 plus 40% of every dollar over \$50000.
Over \$80000	\$18000 plus 60% of every dollar over \$80000.

Situation Two.

To deliver a parcel from town A to town B a company charges \$7·50 for the first kilogram, or part thereof, and then a further \$2·50 per kilogram, or part thereof, after that. Thus a parcel weighing 0·56 kg will cost \$7·50, a parcel weighing 2·4 kilograms will cost \$12·50 (= \$7·50 + 2 × \$2·50), a parcel weighing 5·1 kilograms will cost \$20 (= \$7·50 + 5 × \$2·50) etc.

Situation Three.

John has bought a new racing bike and has sold his old one to Peter. Peter lives 12 kilometres from John's house, along an almost straight road. John rides the bike to Peter's house, stays there for 30 minutes having a chat with Peter, and then walks back to his own house. He cycles at a steady 12 km per hour and walks at a steady 6 km per hour.



Piecewise defined relationships.

The three situations on the previous page (and five of the six graphs) involved circumstances in which the relationship between two variables involved a number of different linear relationships. Which relationship applied depended on where on the horizontal axis we were. In this way different rules applied for different *pieces* of the horizontal axis. Such relationships are said to be **piecewise defined**.

One of the six graphs on the previous page is <u>not</u> a piecewise defined linear graph – which one?

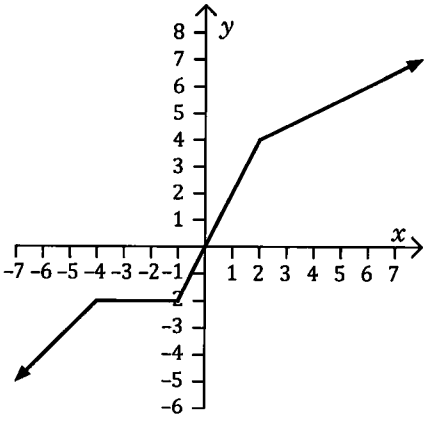
Two of the six graphs on the previous page are also called **step graphs**, which two?

Consider for example the graph on the right.

In this case, for x less than -4 one rule applies, for x from -4 to -1 another rule applies etc.

We write this as:

For	$x < -4$	$y = x + 2$
for	$-4 \le x < -1$	$y = -2$
for	$-1 \le x < 2$	$y = 2x$
for	$x \ge 2$	$y = 0{\cdot}5x + 3$

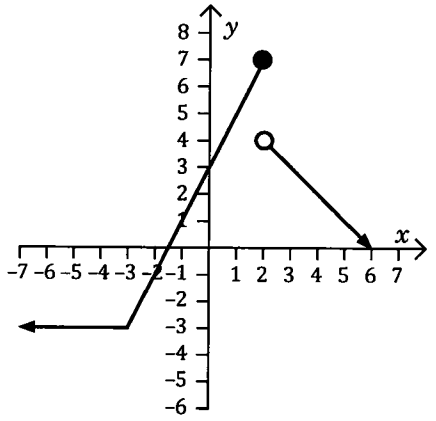


(Note that whilst in the above listing we have attached the $x = -4$ value to the $y = -2$ rule it could equally well have been attached to the $y = x + 2$ rule. Similarly the $x = -1$ and $x = 2$ values could be differently "attached".)

In the piecewise defined relationship shown on the right the filled circle shows where the value for $x = 2$ **is**, and the open circle shows where the value for $x = 2$ **is not**. (Did you notice this aspect in one of the situations on the previous page?)

In this case

For	$x < -3$	$y = -3$
for	$-3 \le x \le 2$	$y = 2x + 3$
for	$x > 2$	$y = -x + 6$

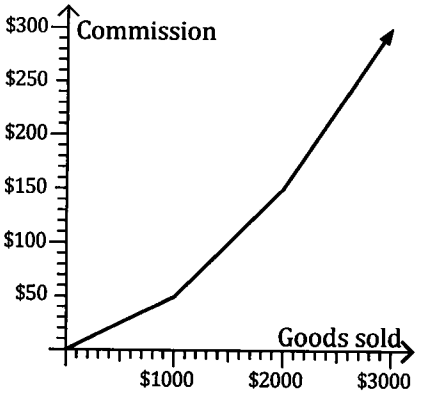


The above examples involved abstract lines without any real context attached but, as the first three situations demonstrated, piecewise defined relationships do occur in real life.

For example consider the situation of a company paying commission for sales achieved by its sales people according to the following rules:

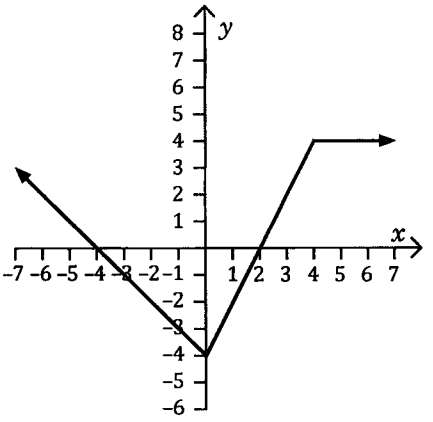
Sales	Commission
\$0 → \$1 000	5% of sales
\$1 000 → \$2 000	\$50 + 10% of each \$1 over \$1 000
\$2 000 and over.	\$150 + 15% of each \$1 over \$2 000

The graph of this situation is shown on the right.



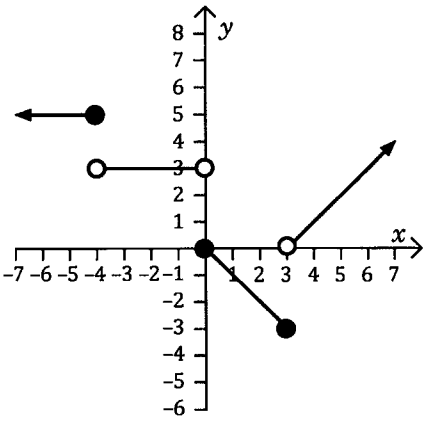
Exercise 9A

1. Copy and complete the following statements for the piecewise defined function shown on the right.



For $x < 0$ y =
for $0 \leq x < 4$ y =
for $x \geq 4$ y =

2. Copy and complete the following statements for the piecewise defined function shown on the right.



For $x \leq -4$ y =
for y =
for y =
for y =

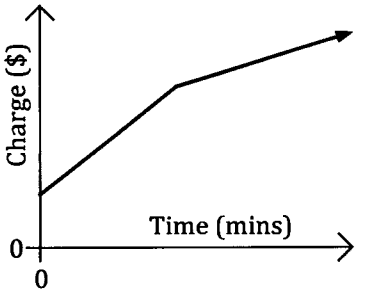
3. Draw the following piecewise defined function:

For $x \leq -5$ $y = -x - 5$
for $-5 < x \leq 3$ $y = x + 5$
for $x > 3$ $y = 8$

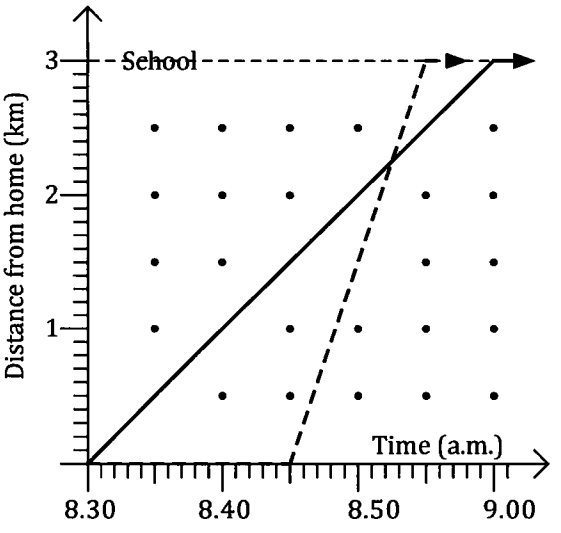
4. Draw the following piecewise defined function:

For $x \leq 0$ $y = -3$
for $0 < x < 4$ $y = 2x$
for $x = 4$ $y = 10$
for $x > 4$ $y = -x + 7$

5. The fare charged by a taxi company depended upon the number of minutes the journey lasted. The graph shown sketched on the right shows the charge in dollars graphed against the time of the journey in minutes. Write a few sentences describing your interpretation of the situation based on the information given by the graph

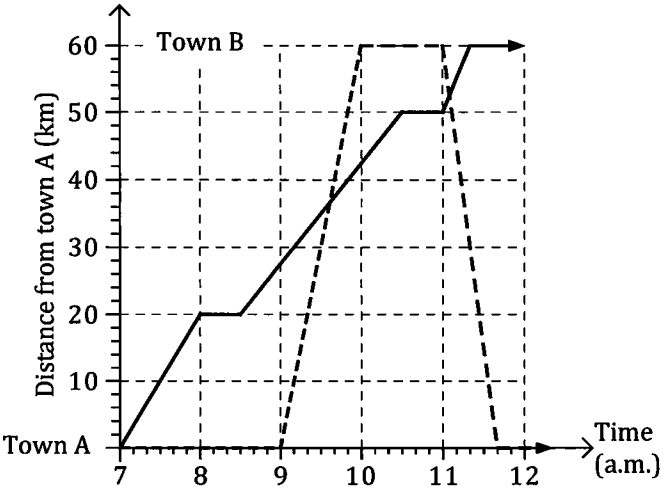


6. The distance time graph on the right is for two brothers journeying from home to the same school. One brother leaves early and walks to school and the other leaves later and cycles.



   (a) Does the broken line " - - - - " represent the journey of the walker or the cyclist?
   (b) Estimate the time when the cyclist passes the walker.
   (c) How many minutes did the walker take to walk to school?
   (d) What was the steady speed maintained by the walker during his walk?
   (e) How many minutes did the cyclist take to ride to school?
   (f) What was the steady speed maintained by the cyclist during his ride?

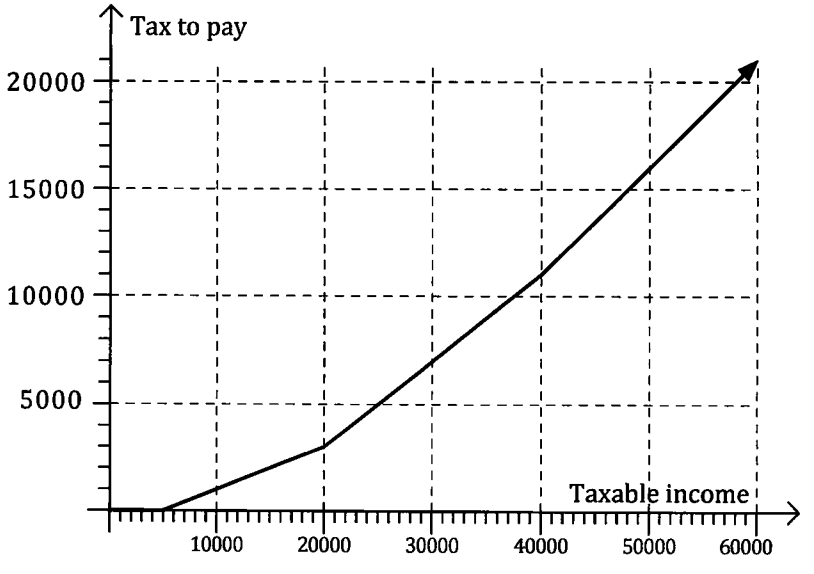
7. The distance time graph shown on the right is for the motion of a cyclist travelling from town A to town B, 60 km away, and a delivery truck making the round trip from A to B and back to A again.



   (a) When did the cyclist leave town A?
   (b) When did the cyclist reach town B?
   (c) The cyclist stopped twice for a rest. How long was each stop ?
   (d) What speed did the cyclist maintain
      (i) prior to the first stop,
      (ii) between the two stops,
      (iii) after the second stop?
   (e) What speed did the delivery truck maintain
      (i) from town A to B,
      (ii) from town B back to A?
   (f) Estimate the time and distance from A of the place where the delivery truck passed the cyclist when they were both travelling towards B.
   (g) Estimate the time and distance from A of the place where the delivery truck passed the cyclist when the truck was returning to A.

8. A long straight road links three towns A, B and C with B between A and C. From town A it is 130 km to B and a further 140 km to C. A truck leaves A at 8 a.m. and travels to B. For the first half hour the truck maintains a steady speed of just 60 km/h due to speed restrictions. After this the truck is able to maintain a higher speed and arrives in town B at 9.30 a.m. Unloading and loading in town B takes 1 hour and then the truck travels on to C maintaining a steady 80 km/h for this part of the journey.
   A car leaves A at 9 a.m. that same morning and travels directly to C. Subject to the same speed restrictions it too maintains a steady 60 km/h for the first half hour. After this first half hour the car then maintains a steady 100 km/h all the way to town C.
   Draw a distance time graph for this situation and use your graph to answer the following questions:
   (a) When does each vehicle reach town C?
   (b) What steady speed did the truck maintain from 8.30 a.m. to 9.30 a.m.?
   (c) What was the average speed of the truck from A to B? (to nearest km/h.)
   (d) When and where did the car pass the truck?

9. The graph on the right shows a typical income tax system. The higher an individual's taxable income the more tax the person must pay.
   Use the graph to estimate the tax payable by someone with a taxable income of
   (a) \$30 000,
   (b) \$40 000,
   (c) \$48 000,
   (d) \$3 000.

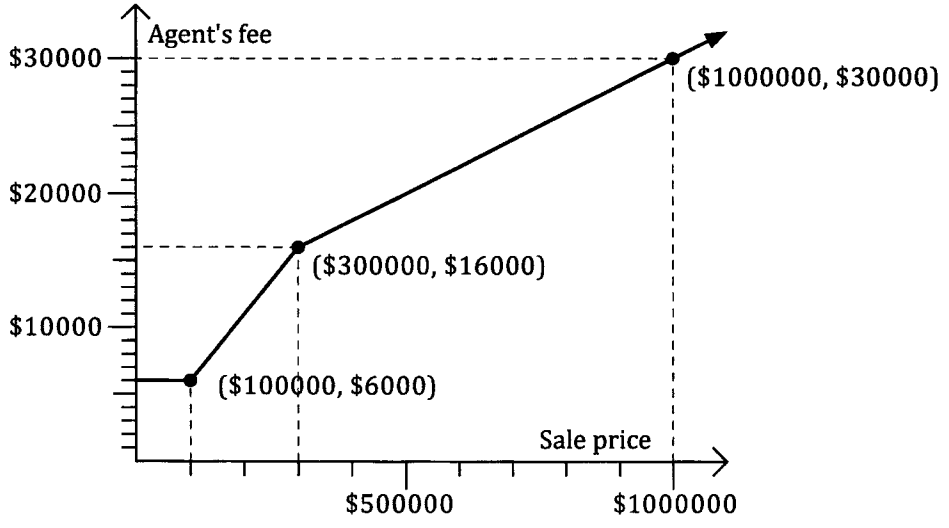


   (e) If a person has to pay tax of \$20 000 what is their taxable income?

10. Draw the graph for the commission paid by a company to its sales people if payments are made according to the following rules:

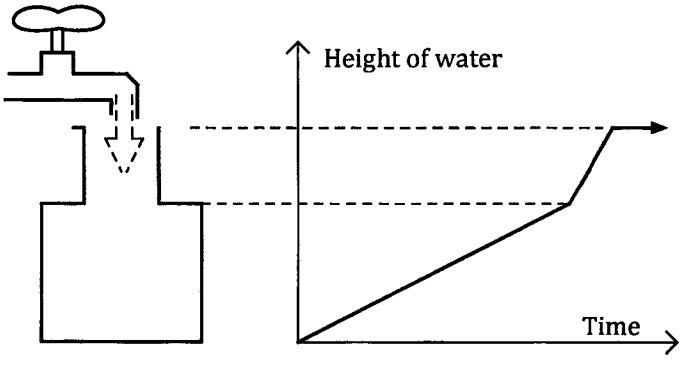
Sales	Commission
\$0 → \$5 000	4% of sales
\$5 000 → \$10 000	\$200 + 6% of each \$1 over \$5 000
Over \$10 000.	\$500 + 10% of each \$1 over \$10 000

11. When a real estate agent arranges for the sale of a house the owner of the house pays the agent a fee, often based on the amount the house sells for.
Let us suppose that one agent's fee structure is as shown in the graph below.

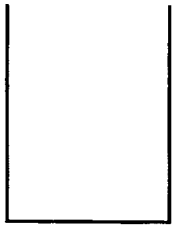


Express this piecewise defined relationship as follows with the blanks completed.
For the first $_____ of the sale price the agent's fee is a fixed $_____.
From $_____ to $_____ the fee is $_____ plus _____% of the amount over $_____.
From $_____ and over the fee is $_____ plus _____% of the amount over $_____.

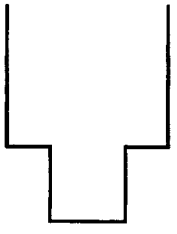
12. Suppose water is flowing at a constant rate into a container. For a container shaped as shown on the right the graph of the height of the water level plotted against time would be as shown.
Sketch the graph of the height of the water level against time for each of the following containers.



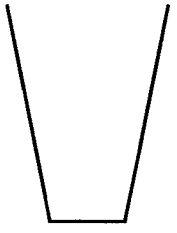
(a)



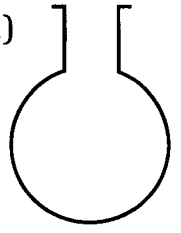
(b)



(c)



(d)



Miscellaneous Exercise Nine.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

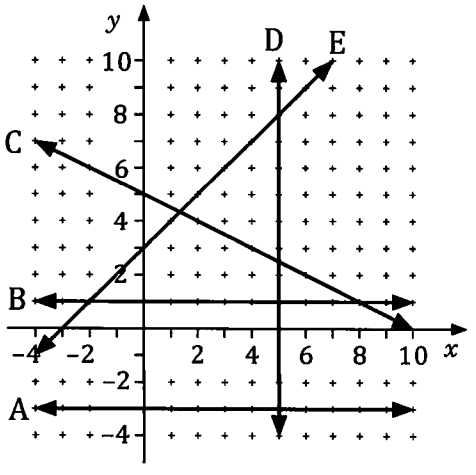
1. Find the rule that exists between P and t given the following table.

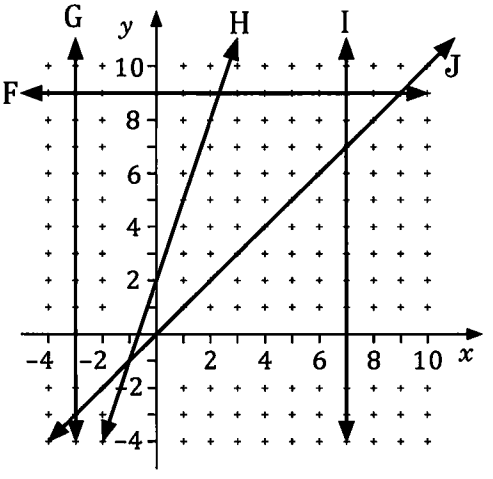
t	2	3	4	5	6	7
P	1	4	7	10	13	16

2. Given that the relationship between x and y is linear find the values of $a, b, c, \ldots g$.

x	0	1	2	3	4	5	6	...	f	g
y	a	b	c	14	d	24	e	...	54	494

3. Write the equations of each of the lines A to J shown in the graphs below.





4. A company determines that the cost, $\$C$, for the production of x radios of a particular type is given by: $C = 5200 + 16x$

   (a) Interpret the 5200 and 16 in this equation in the context of this question.

   Find the mean cost per radio when (b) 100 radios are produced,
   (c) 500 radios are produced,
   (d) 1 000 radios are produced.

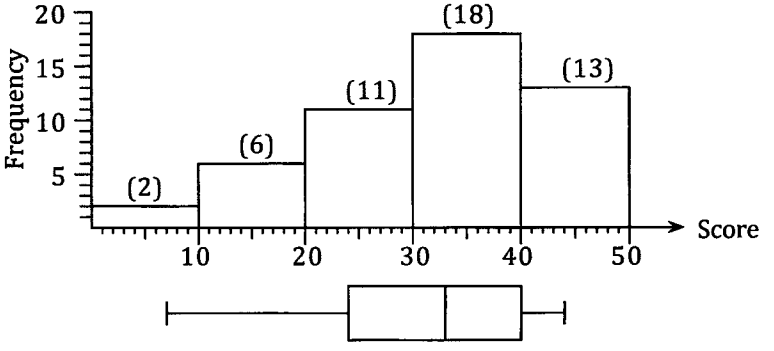
5. The ratio of year eight students in a school to non year eight students in the school is 7 : 25. If there are 960 students in the school altogether how many year eight students are there in the school?

6. What number gives you the same answer when you add sixteen to it as when you multiply it by 5?

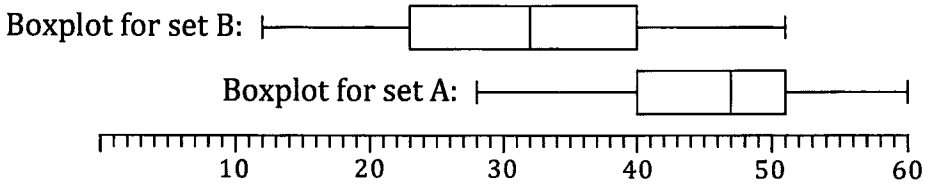
7. I think of a number, double it, add five, multiply the result by four, take away the number I first thought of and end up with sixty two. Find the number first thought of.

8. A set of numbers consists of 2 fours, 8 fives, 11 sixes, 9 sevens and a number of eights. If the mean of the set is 6·2 determine the number of eights in the set.

9. The histogram and the box plot for a data set are shown on the right. Determine
   (a) the median,
   (b) the range,
   (c) the interquartile range,
   (d) the modal class.
   (e) Use your calculator to determine an estimate for the mean.



10. Two sets of students sat the same test and the boxplots of their marks are shown below.



   (a) What percentage of students in set A exceeded the highest mark obtained by students in set B?
   (b) Which two features of the boxplots suggest that the marks in set B were more variable (i.e. more spread out) than those in set A?

   If the top 25% of the students in set B, as defined by the test results, were all moved to set A how would:
   (c) the median mark of those left in set B compare with that of set B before the move?
   (d) the range of the scores of those left in set B compare with that of set B before the move?
   (e) the range of scores of the new set A compare with the range of the scores of set A before the move?
   (f) the interquartile range of the new set A compare with the interquartile range of the scores of set A before the move?